Write

$$\alpha \equiv \tfrac{1}{2}(x_{2n} + y_{2n})$$

Then $\alpha - x \geq \alpha - x_{2n} - |x_{2n} - x| \geq \tfrac{1}{2}(y_{2n} - x_{2n}) - (2n)^{-1} > 0$

Also, $y - \alpha \geq y_{2n} - \alpha - |y_{2n} - y| \geq \tfrac{1}{2}(y_{2n} - x_{2n}) - (2n)^{-1} > 0$

Therefore $x < \alpha < y$.

As a corollary, for each x in $\mathbb{R}$ and r in $\mathbb{R}^+$ there exists α in Q with $|x - \alpha| < r$. Here is another corollary.

Proposition 7 *If $x_1, \ldots, x_n$ are real numbers with $x_1 + \cdots + x_n > 0$, then $x_i > 0$ for some i $(1 \leq i \leq n)$.*

Proof By Lemma 4 there exists a rational number α with $0 < \alpha < x_1 + \cdots + x_n$. Let a_i $(1 \leq i \leq n)$ be a rational number with

$$|x_i - a_i| < (2n)^{-1}\alpha$$

Then
$$\sum_{i=1}^{n} a_i \geq \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} |x_i - a_i| > \tfrac{1}{2}\alpha$$

Therefore $a_i > (2n)^{-1}\alpha$ for some i. It follows that

$$x_i \geq a_i - |x_i - a_i| > 0$$

Corollary *If x, y, and z are real numbers with $y < z$, then either $x < z$ or $x > y$.*

Proof Since $z - x + x - y = z - y > 0$, either $z - x > 0$ or $x - y > 0$, by Proposition 7.

The next lemma gives an extremely useful method for proving inequalities of the form $x \leq y$.

Lemma 5 *Let x and y be real numbers such that the assumption $x > y$ implies that $0 = 1$. Then $x \leq y$.*

Proof Without loss of generality, we take $y = 0$. For each n in Z^+, either $x_n \leq n^{-1}$ or $x_n > n^{-1}$. The case $x_n > n^{-1}$ is ruled out, since it implies $x > 0$. Therefore $-x_n \geq -n^{-1}$, for all n, so that $-x \geq 0$.

Theorem 1 *Let $\{a_n\}$ be a sequence of real numbers. Let x_0 and y_0 be real numbers, $x_0 < y_0$. Then there exists a real number x with*

$$x_0 \leq x \leq y_0 \tag{2.22}$$

and

$$x \neq a_n \qquad (n \in Z^+) \tag{2.23}$$

Proof *We construct by induction sequences $\{x_n\}$ and $\{y_n\}$ of rational numbers such that*

(2.24)
(i) $x_0 \leq x_n \leq x_m < y_m \leq y_n \leq y_0 \qquad (m \geq n \geq 1)$
(ii) $x_n > a_n$ or $y_n < a_n \qquad (n \geq 1)$
(iii) $y_n - x_n < n^{-1} \qquad (n \geq 1)$

Assume that $n \geq 1$ and that $x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}$ have been constructed. Either $a_n > x_{n-1}$ or $a_n < y_{n-1}$. In case $a_n > x_{n-1}$, let x_n be any rational number with $x_{n-1} < x_n < \min\{a_n, y_{n-1}\}$, and let y_n be any rational number with $x_n < y_n < \min\{a_n, y_{n-1}, x_n + n^{-1}\}$. Then the relevant inequalities are satisfied. In case $a_n < y_{n-1}$, let y_n be any rational number with $\max\{a_n, x_{n-1}\} < y_n < y_{n-1}$, and x_n any rational number with $\max\{a_n, x_{n-1}, y_n - n^{-1}\} < x_n < y_n$. Again, the relevant inequalities are satisfied. This completes the induction.

From (i) and (iii) it follows that

$$|x_m - x_n| = x_m - x_n < y_n - x_n < n^{-1} \qquad (m \geq n)$$

Similarly $|y_m - y_n| < n^{-1}$ for $m \geq n$. Therefore $x \equiv \{x_n\}$ and $y \equiv \{y_n\}$ are real numbers. By (iii), they are equal. By (i), $x_n \leq x$ and $y_n \geq y$ for all n. If $a_n < x_n$ then $a_n < x$, so $a_n \neq x$. If $a_n > y_n$ then $a_n > y = x$, so $a_n \neq x$. Thus x satisfies (2.22) and (2.23).

Theorem 1 is the famous theorem of Cantor, that the real numbers are uncountable. The proof is essentially Cantor's "diagonal" proof. Both Cantor's theorem and his method of proof are of great importance.

The time has come to consider some counterexamples. Let $\{n_k\}$ be a sequence of integers, each of which is either 0 or 1, for which we are unable to prove either that $n_k = 1$ for some k or that $n_k = 0$ for all k. This corresponds to what Brouwer calls "a fugitive property of the natural numbers." Such a sequence can be defined, for example, as